QUASI-CONFORMAL DEFORMATIONS OF NONLINEARIZABLE GERMS

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Abstract. Let $f(z) = e^{2\pi i\alpha}z + O(z^2), \alpha \in \mathbb{R}$ be a germ of holomorphic diffeomorphism in $\mathbb{C}$. For α rational and f of infinite order, the space of conformal conjugacy classes of germs topologically conjugate to f is parametrized by the Ecalle-Voronin invariants (and in particular is infinite-dimensional). When α is irrational and f is nonlinearizable it is not known whether f admits quasi-conformal deformations. We show that if f has a sequence of repelling periodic orbits converging to the fixed point then f embeds into an infinite-dimensional family of quasi-conformally conjugate germs no two of which are conformally conjugate.

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Contents

1. Introduction

Let $f(z) = e^{2\pi i\alpha}z + O(z^2), \alpha \in \mathbb{R}/\mathbb{Z}$ be a germ of holomorphic diffeomorphism fixing the origin in $\mathbb{C}$. We consider the question of when f admits quasi-conformal deformations, i.e. when do there exist germs g which are quasi-conformally but not conformally conjugate to f? If f is *linearizable* (i.e. analytically conjugate to the rigid rotation $R_\alpha(z) = e^{2\pi i\alpha}z$) then any germ topologically conjugate to f is linearizable. In the *nondegenerate parabolic* case (i.e. $\alpha = p/q \in \mathbb{Q}, f^q \neq id$), the quasi-conformal conjugacy class of f contains an infinite dimensional family of conformal conjugacy classes parametrized by the Ecalle-Voronin invariants ([Eca75], [Vor81]). In the *irrationally indifferent nonlinearizable* case (α irrational, f not linearizable), it seems to be unknown whether quasi-conformal deformations are possible. We show the following:

Theorem 1.1. *Let f be an irrationally indifferent nonlinearizable germ with a sequence of repelling periodic orbits accumulating the origin. Then there is a family of quasi-conformal maps $\{h_\Lambda\}\Lambda \in \mathcal{M}$ parametrized by sequences $\mathcal{M} = \{(\lambda_n)_{n\geq 0}$:*

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$\lambda_n \in \mathbb{C}, |\lambda_n| > 1\}$ such that all conjugates of $g_\Lambda = h_\Lambda \circ f \circ h_\Lambda^{-1}$ are holomorphic, and $g_{\Lambda_1}, g_{\Lambda_2}$ are conformally conjugate if and only if all but finitely many terms of Λ_1 and Λ_2 agree. For fixed z, $h_\Lambda(z)$ depends holomorphically on each λ_n.

The proof proceeds as follows: given $\Lambda = (\lambda_n)$ the germ g_Λ is obtained by quasi-conformally deforming f near its periodic orbits. Each periodic orbit is attracting for f^{-1}, and it is possible to construct an f-invariant Beltrami differential on each basin of attraction such that the quasi-conformal map h_Λ rectifying the Beltrami differential conjugates f to a holomorphic germ g_Λ having multipliers λ_n at the periodic orbits. The multipliers at periodic orbits being invariant under conformal conjugacies, the conclusion of the theorem follows.

Examples of germs satisfying the hypothesis of the theorem are germs of rational maps of degree d with α satisfying the Cremer condition of degree d (see for example Milnor [Mil99], Ch. 8)

$$\limsup \frac{\log \log q_{n+1}}{q_n} > \log d$$

(where (p_n/q_n) are the continued fraction convergents of α) which ensures that the fixed point is accumulated by periodic orbits (only finitely many of which can be repelling for a rational map).

Perez-Marco has shown ([PM97]) for any nonlinearizable germ f the existence of a unique monotone one-parameter family $(K_t)_{t>0}$ of full, totally invariant continua called *hedgehogs* containing the fixed point. In [Bis09] it is proved that any conformal mapping in a neighbourhood of a hedgehog K of a germ f_1 mapping K to a hedgehog of a germ f_2 necessarily conjugates f_1 to f_2. As a corollary of Theorem 1.1 we have

Corollary 1.2. *There exists a holomorphic motion $\phi : \mathbb{D}^* \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of $\hat{\mathbb{C}}$ over $\mathbb{D}^*$ and a hedgehog K such that all the sets $\phi(t, K)$ are hedgehogs, all of which are quasi-conformal images of K, but for $s \neq t$, $\phi(s, K)$ cannot be conformally mapped to $\phi(t, K)$.*

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2. Deformations.

We fix a germ $f(z) = e^{2\pi i \alpha} z + O(z^2), \alpha \in \mathbb{R} - \mathbb{Q}$ and a neighbourhood U of the origin such that f and f^{-1} are univalent on a neighbourhood of $\overline{U}$. By a *periodic orbit* or *cycle* of f of order $q \geq 1$ we mean a finite set $\mathcal{O} = \{z_1, \ldots, z_q\} \subset U$ such that $f(z_i) = z_{i+1}, 1 \leq i \leq q-1$ and $f(z_q) = z_1$. The *multiplier* at the periodic orbit is defined to be $\lambda = f'(z_1) f'(z_2) \ldots f'(z_q) = (f^q)'(z_i)$. The periodic orbit is called *attracting*, *indifferent* and *repelling* according as $|\lambda| < 1, |\lambda| = 1$ and $|\lambda| > 1$ respectively. A periodic orbit for f of multiplier λ is a periodic orbit for f^{-1} of multiplier λ^{-1}. The basin of attraction of an attracting periodic cycle is defined by $\mathcal{A}(\mathcal{O}, f) := \{z \in U : f^n(z) \to \mathcal{O} \text{ as } n \to +\infty\}$.

We observe that f can have only finitely many cycles of a given order q in U (by the uniqueness principle). Since f is asymptotic to an irrational rotation, i.e. $f(z)/z \to e^{2\pi i\alpha}$ as $z \to 0$, it follows that if f has small cycles then the orders of the cycles must go to infinity.

2.1. **Deforming repelling periodic orbits.** Given a repelling periodic orbit $\mathcal{O}$ of f with multiplier λ and $|\lambda'| > 1$ we deform the multiplier of f quasi-conformally from λ to λ' by constructing a corresponding f-invariant Beltrami differential $\mu = \mu(\mathcal{O}, \lambda, \lambda')$ on the basin of attraction $\mathcal{A}(\mathcal{O}, f^{-1})$ as follows:

By Köenigs linearization theorem (see [Mil99], Ch. 6) for any repelling periodic orbit $\mathcal{O}$ of f with multiplier λ there exists a unique holomorphic map ϕ defined on a neighbourhood of $\mathcal{O}$ such that $\phi(z_i) = 0, \phi'(z_i) = 1, i = 1, \dots, n$ and $\phi(f^q(z)) = \lambda\phi_\lambda(z)$. The conformal isomorphism $L : \mathbb{C}^* \to \mathbb{C}/\mathbb{Z}, w \mapsto \xi = \frac{1}{2\pi i}\log w$ conjugates the linear map $w \mapsto \lambda w$ on $\mathbb{C}^*$ to the translation $\xi \mapsto \xi + \tau$ on $\mathbb{C}/\mathbb{Z}$ where $\tau = L(\lambda)$ and $\operatorname{Im}\tau < 0$.

Let $\tau' = L(\lambda')$ and let K be the real linear map on $\mathbb{C}$ defined by $K(1) = 1, K(\tau) = \tau'$. Then K commutes with the translation by one and hence gives a quasi-conformal orientation preserving (since $\operatorname{Im}\tau, \operatorname{Im}\tau' < 0$) homeomorphism $\tilde{K} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$. The Beltrami differential of $\tilde{K}$ is constant and invariant under translations of $\mathbb{C}/\mathbb{Z}$, and $\tilde{K}$ conjugates the translation $\xi \mapsto \xi + \tau$ on $\mathbb{C}/\mathbb{Z}$ to $\xi \mapsto \xi + \tau'$.

We let μ be the Beltrami differential of $\tilde{K} \circ L \circ \phi_\lambda$ restricted to a small neighbourhood D of $z_1 \in \mathcal{O}$. The map $\tilde{K} \circ \phi_\lambda$ conjugates f^q to the translation $\xi \mapsto \xi + \tau'$. So at points $z, z' = f^q(z)$ in D, μ satisfies the invariance condition

$$\mu(z')\overline{(f^q)'(z)} = \mu(z)(f^q)'(z)$$

So μ extended to the neighbourhood $V = D \cup f(D) \cup \dots \cup f^{q-1}(D)$ of $\mathcal{O}$ by putting

$$\mu(f^j(z)) = \mu(z)\frac{(f^j)'(z)}{\overline{(f^j)'(z)}}$$

for $z \in D, j = 1, \dots, q-1$ is an f-invariant Beltrami differential. Similarly the above equation allows us to extend μ to an f-invariant Beltrami differential $\mu(\mathcal{O}, \lambda, \lambda')$ on $\mathcal{A}(\mathcal{O}, f^{-1}) = \cup_{n \geq 1} f^n(V)$.

Lemma 2.1. *Any quasi-conformal homeomorphism h with Beltrami coefficient equal to $\mu = \mu(\mathcal{O}, \lambda, \lambda')$ on a neighbourhood of $\mathcal{O}$ conjugates f to a map $g = h \circ f \circ h^{-1}$ with a periodic orbit $h(\mathcal{O})$ of multiplier λ'. The dependence of $\mu(\mathcal{O}, \lambda, \lambda')$ on λ' is holomorphic.*

Proof: For $i = 1, \dots, q$ we let ψ_i be the branch of ϕ^{-1} sending 0 to z_i. By construction the map k defined by $k = \psi_i \circ L^{-1} \circ \tilde{K} \circ L \circ \phi(z)$ for z in a neighbourhood of z_i has Beltrami coefficient equal to μ and conjugates f on a neighbourhood of $\mathcal{O}$ to a holomorphic map $f_1 = k \circ f \circ k^{-1}$ with periodic orbit $k(\mathcal{O})$ and multiplier λ'. Since h and k have the same Beltrami coefficient the map $h \circ k^{-1}$ is holomorphic, and conjugates f_1 to g, so the multipliers of f_1 and g are equal. The Beltrami differential of $\tilde{K}$ is constant equal to $\frac{-\log(\lambda'/\lambda)}{2\log|\lambda| + \log(\lambda'/\lambda)}$ which depends holomorphically on λ', so the Beltrami differential $\mu(\mathcal{O}, \lambda, \lambda')$ of $\tilde{K} \circ L \circ \phi_\lambda$ depends holomorphically on λ'. $\diamond$

2.2. **Deforming germs with small cycles.** We use the Beltrami differentials $\mu(\mathcal{O}, \lambda, \lambda')$ to deform a germ with small cycles as follows:

Proof of Theorem 1.1: Let f be a germ with a sequence of repelling small cycles of orders (q_n) (which we may assume to be strictly increasing). The basins of attraction $\mathcal{A}(\mathcal{O}, f^{-1})$ of distinct repelling cycles of f are disjoint, so for any $\Lambda = (\lambda_n) \in \mathcal{M}$ we can define an f-invariant Beltrami differential μ_Λ on U by putting $\mu_\Lambda(z) = \mu(\mathcal{O}, \lambda, \lambda_n)(z)$ when z belongs to $\mathcal{A}(\mathcal{O}, f^{-1})$ for a cycle $\mathcal{O}$ of order q_n and multiplier λ, and $\mu_\Lambda(z) = 0$ otherwise. Let h_Λ be the unique quasi-conformal homeomorphism with Beltrami coefficient μ_Λ fixing $0, 1, \infty$ given by the Measurable Riemann Mapping Theorem. Then $g_\Lambda = h_\Lambda \circ f \circ h_\Lambda^{-1}$ is a holomorphic germ fixing the origin with small cycles. By Naishul's theorem [Nai83] the multiplier at an indifferent fixed point is a topological conjugacy invariant so $g'_\Lambda(0) = e^{2\pi i \alpha}$. By Lemma 2.1, the multiplier of g_Λ at *all* its repelling cycles of order q_n is equal to λ_n.

If ϕ is a holomorphic germ conjugating two such germs $g_{\Lambda_1}, g_{\Lambda_2}$, ϕ must take all repelling cycles of g_{Λ_1} of order q_n (in its domain) to repelling cycles of g_{Λ_2} of order q_n and preserve multipliers, so the sequences of multipliers Λ_1, Λ_2 must agree for all but finitely many terms. It follows from Lemma 2.1 that μ_Λ depends holomorphically on each λ_n, hence by the Measurable Riemann Mapping Theorem so does $h_\Lambda(z)$ for fixed z. $\diamond$

Proof of Corollary 1.2: Let f be as above and $K \subset U$ be a hedgehog of f. For $t \in \mathbb{D}^*$ we let Λ_t be the constant sequence $(\lambda_n = 1/t)$. Then μ_{Λ_t} depends holomorphically on t, and $\phi : (t, z) \mapsto h_{\Lambda_t}(z)$ gives the required holomorphic motion. $\diamond$

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